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Influence of predator incited fear and interference competition in the dynamics of prey-predator system where the prey species are protected in a reserved area

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ABSTRACT

In this article, we contemplated a prey-predator model where the dwelling place of prey species is partitioned into two areas, namely reserved area and unreserved area. Both the prey and predator species are allowed in unreserved area to access their living resources; whereas reserved area is protected only for prey species and predators are not allowed by any means in this area. At the unreserved zone, predation is allowed and prey species are in fear of predation by the predators. As a consequence, it is assumed that prey species in unreserved zone shows many anti-predator behaviour and defense mechanism (due to fear of predation), which create the shortage of food and compels the predators to engage into a clash among themselves. After completion of model construction, the persistence of all the species and well-posedness of the model system have been studied. The existence and stability criterion of all the equilibrium points have been found. Furthermore, bifurcation analysis is done and direction of Hopf-bifurcation is detected using center manifold theory. Through model analysis and rigorous numerical simulation, it is observed that cost of fear and interference competition among the predators cooperatively determine the eco-dynamics of the model system in a complex manner. Finally, we have plotted various 2-parameter bifurcation diagrams near the hopf-point to explore the role of different parameters involved in the model system.

Key words : *Predator incited fear, Interference competition, Hopf-bifurcation, Fold bifurcation*

Introduction

In view of ecological extinction of several species, the necessity of the creation of reserved area for some species (specially those who are at the verge of eco-extinction) is increasing day by day. Many species in ecology are at the skirt of eradication due to diverse man-made external forces like unscientific harvesting, poaching, over exploitation, excessive environmental pollution, misuse of resources, loss of dwelling place etc. and many internal forces like excessive predation, unprecedented environmental fluctuation and abnormal change in climate etc. [1–4]. These external and internal forces jointly or separately compel a number of species to migrate from one region to another region. Many creatures such as monarch butterfly, blue whale, sandhill cranes, humpback whales, wildebeest, gray whales, hummingbirds, Canada goose, pronghorn, snow geese migrate from one place to another place in search of food & water, mates, new habitat to save themselves from cruelty of climate conditions and natural calamities. However, nowadays, several eco-friendly agencies, organizations and governments of different countries have adopted various strategies such as imposing new laws to restrict hunting and unnecessary killing of animals, making several reserved areas like national park, wildlife sanctuaries, etc., to protect species specially that are at the verge of extinction from our eco-environment. In this context, the role of refuges and importance of reserved zones for prey have been nicely pointed out by several researchers [5–12] through appropriate mathematical modeling and meticulous numerical simulations. Most of these articles confirmed the fact that refuge of prey helps the system dynamics to be stable, i.e. refuge of prey has stabilizing effect on an eco-dynamical system. However, Collings [5] studied a mite-fruit interaction model with refuge of the prey species and manifested that, in case of refuge of prey, both the prey and predator population biomass reduced drastically which ultimately destabilizes the system dynamics. Chattopadhyay et al. [6] conferred a predator-prey model with some cover on the prey species and demonstrated an important fact that global stability of the interior equilibrium point of an ecological model system does not always signify the permanence of the system. In the aquatic environment, researchers [13–17] have critically analyzed the importance of reserve zone in conservation of fish species by protecting them from their over exploitation. Dubey et al. [13] studied a fishery model in aquatic environment and reveals the fact that the density of fish population at equilibrium level can be maintained appropriately even under the continuous pressure of over exploitation. Several other research works [18–24] have been focused to study the ecological system considering reserved & unreserved zone from different perspective, which enormously helps eco-policy makers to implement new laws and/or policies to save many endangered species.

On the other hand, many research works have been carried out in ecological modeling to study and explore the dynamics of prey-predator interaction by incorporating “trait-mediated effects” [25]. A good number of studies [26–30] revealed the fact that density mediated “deadly impacts” influence the wellness of prey populace; while few other articles [31–35] pointed out that many trait-mediated “interceded impacts” (like predators incited fear factor/ fear level) can drastically change the topology of the physical environment of the interacting individuals by deflecting the scavenge activities and reshaping of psychology of the prey species. These characteristic secondary impacts, especially the fear influence, may change depending on the situation. Many instances may be given in this respect; for example, a salamander reduces its rummaging activity due to a perceived danger of predation by a garter snake [36]; birds leave their nest in react to predator’s sound [31]; due to predator signals, the live bearing guppy reduces their swimming efficiency during the advanced time of development and avoids predation by shortening brood detainment period [37]. These backhanded impacts can be deadly for the adult prey species which consequently decrease their breeding success [38, 39]. Blind reduction in foraging activity to avoid predation hazard, sometimes forces the prey to adopt starvation mechanism which causes as a drop in prey’s growth rate in long run [40]. The sub-optimal quality of new habitat (which has been adopted by prey species through extreme hurry due to pressure of fear of predation) such as lack of proper nourishment becomes disadvantageous to the survival of mature prey species [31].

In the year 2011, Zanette et al. [41] performed a field experiment and observed that the quantity of posterity of song-sparrow decreased upto 40% due to predator incited fear factor. Each of these experimental shreds of evidence along with theoretical results suggest that the various backhanded “trait-mediated effects” should be incorporated in a prey-predator model along with direct predation. In these point of view, Wang et al. [32] was first to incorporate the concept of fear level induced by the predators in a prey-predator model and ascertained that fear level plays as a different stabilising scenarios when predator species devours the prey individual conforming to Holling type-II functional scheme. Recently, many researchers [33, 42–53] have nicely and crucially incorporated the fear factor (fear occurs due to predation by predator) along with other crucial factors to discuss the dynamics of different eco-epidemic and ecological models.

But, till date, we have not noticed any study of predator-prey model incorporating predator incited fear where the prey species are protected in a reserved area. Therefore, in this article, we have initiated a predator-prey model with reserved area for prey species incorporating predator incited fear and intra-specific competition among predators to explore the answer of the following questions:

(a) How / When the non-lethal effect such as “fear effect” regulates the dynamics of the system ?
(b) Is there any inter linkage between intra-specific competition and predator incited fear among predators?
(c) How intra-specific competition and the predator incited fear among predators regulate dynamics of the system?

For the current study the above mentioned questions are motivated to us.

The rest of this article has been arranged in several sections. Section-2 deals with the construction of the mathematical model system. Next, boundedness, Positivity and persistence have been described in Section-3. Section-4 demonstrates local stability analysis and Section-5 reports about the local bifurcation analysis (LP and Hopf bifurcation).

In Section-6, we performed extensive numerical simulations & discussed the results. finally Section-7 is ended up with conclusion .

Mathematical construction of the model system

Here, we consider a prey-predator model where the habitat of prey species is partitioned into two areas, namely reserved area and unreserved area. Both the prey and predator species are allowed in unreserved area to access their living resources; whereas reserved area is protected only for prey species and predators are not allowed by any means in this area. At any time t, let x(t); y(t) and z(t) be the prey population density in unreserved area, the prey population density in reserved area and the predator population density. The mathematical formulation of the model system is constructed step by step in consideration of several assumptions as follows:

(i) Due to the absence of predator, the prey species of unreserved area grows logistic scheme at a rate r_1 with natural mortality rate d_1. In the reserved zone prey species grows logistic scheme also with intrinsic growth rate r_2 and k is the carrying capacity of reserved zone. Thus, mathematically we can write down the following differential equation

$$\begin{cases} \frac{dx}{dt} = r_1 x - d_1 x, \\ \frac{dy}{dt} = r_2 y\left(1 - \frac{y}{k}\right). \end{cases}$$

(ii) In the unreserved zone, the prey populations x(t) are engaged into a clash due to limitation of resources of livelihood and let m be the intra specific competition rate. Moreover, prey species migrates at a constant rate 1 to save themselves from predator from unreserved zone to the reserved zone; whereas prey species migrates at a constant rate 2 from reserved zone to the unreserved zone in search of food and other resources. Thus the mentioned differential equation (i) reduces to as bellow:

$$\begin{cases} \frac{dx}{dt} = (r_1 - d_1)x - mx^2 - \delta_1 x + \delta_2 y, \\ \frac{dy}{dt} = r_2 y\left(1 - \frac{y}{k}\right) + \delta_1 x - \delta_2 y. \end{cases}$$

(iii) The predator species z(t) consumes the prey species of unreserved zone according to Holling Type-II functional response at a rate c1. Mathematically we can write down it as bellows:

$$\frac{dx}{dt} = (r_1 - d_1)x - mx^2 - \delta_1 x + \delta_2 y - \frac{c_1 xz}{b + x},$$

where b denotes the half-saturation constant for the corresponding prey species

(iv) In the unreserved zone, it is assumed that prey populations are in fear of predation which consequently affect on the growth. Thus, in the mathematical equation, the growth term is multiplied by fear induced function g(f; z).

Hence, the above differential equation can be modified as follows:

$$\frac{dx}{dt} = r_1 g(f, z)x - mx^2 - \delta_1 x + \delta_2 y - \frac{c_1 xz}{b + x} - d_1 x,$$

where f stands for the level of fear exerted from predator species. It is observed that the fear function g(f, z) = 1/(1+fz) satisfies the following assumptions

(a) g(0; z) = 1, i.e. if there is no such fear the growth rate of prey in the unreserved area remains unchanged.
(b) g(f; 0) = 1, i.e. due to the non-appearance of predator the growth rate of prey in the unreserved area remains unchanged.
(c) lim f→∞ g(f, z) = 0, i.e. when the level of fear is too high then the birth rate of prey in the unreserved area decreases and approaches to zero.
(d) lim z→∞ g(f, z) = 0, i.e. When predators are in the amount of large number then the birth rate of prey in the unreserved area decreases and approaches to zero.

(e) ∂g/∂f (f, z) < 0, i.e. with the increment in anti-predator behaviour the growth rate of prey in the unreserved area reduces.

(f) ∂g/∂z (f, z)< 0, i.e. with the increase of predator population density the growth rate of prey in the unreserved area reduces

(v) At the unreserved zone, predation is allowed and prey species are in fear of predation by the predators. As a consequence, it is assumed that prey species has the tendency to leave the unreserved area and/or they show strong anti-predator behaviour and defense mechanism (due to fear of predation), which create the shortage of food and compel the predators to engage into a clash among themselves. Let μ be the rate of competition among the predators, c_2 be the conversion coefficient of predation and d_2 be the natural death rate of predators. Thus we can write the following equation:

$$\frac{dz}{dt} = \frac{c_2xz}{b+x} - \mu z^2 - d_2z.$$

Considering all the above assumptions and combining the mathematical expressions, we get the following model system with initial conditions:

$$\begin{cases} \frac{dx}{dt} = \frac{r_1x}{1+fz} - mx^2 - \delta_1x + \delta_2y - \frac{c_1xz}{b+x} - d_1x, x(0) > 0 \\ \frac{dy}{dt} = r_2y\left(1-\frac{y}{k}\right) + \delta_1x - \delta_2y, y(0) > 0 \\ \frac{dz}{dt} = \frac{c_2xz}{b+x} - \mu z^2 - d_2z, z(0) > 0 \end{cases} \tag{2.1}$$

It is to be noted that here each model parameters are not-negative with $c_1 \geq c_2$.

Mathematical Preliminaries

Positivity of the model system

Theorem 3.1. Every solution of model system (2.1) along with the initial conditions exists in [0, ∞), remains positive ∀ t > 0.

Proof. The system (2.1) can be written as

$$X(t) = M(X(t)), \tag{3.1}$$

$$M(X(t)) = \begin{pmatrix} \frac{r_1x(t)}{1+fz(t)} - m(x(t))^2 - \delta_1x(t) + \delta_2y(t) - \frac{c_1x(t)z(t)}{b+x(t)} - d_1x(t) \\ r_2y(t)\left(1-\frac{y(t)}{k}\right) + \delta_1x(t) - \delta_2y(t) \\ \frac{c_2x(t)z(t)}{b+x(t)} - \mu(z(t))^2 - d_2z(t) \end{pmatrix},$$

where X(t) = col (x(t); y(t); z(t)) and X(0) = col (x_0; y_0; z_0) $\in R^3_+$, and

where $M : R^3 \to R^3_+$ and $M \in C^\infty(R^3)$.

Now, from equation (3.1), it is obvious that M_i(X_i)l_x_i$_{i=0} \geq 0$; for i = 1, 2, 3.

Due to Nagumo [54], the solution of (3.1) with initial conditions $M_0 \in R^3_+$, say M(t) = M(t, M_0) such that $M(t) \in R^3_+$, for all t > 0.

Boundedness

Theorem 3.2. Each solutions of the model system (2.1) of equations are bounded everywhere.

Proof. Here all population density are adding of system (2.1), we get a function, namely W(t) = x(t) + y(t) + z(t).

The derivative of W(t) with respect to time t is

$$\begin{aligned} \frac{dW}{dt} &= \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} \\ &= \frac{r_1x}{1+fz} - mx^2 - \frac{c_1xz}{b+x} - d_1x + r_2y\left(1-\frac{y}{k}\right) + \frac{c_2xz}{b+x} - \mu z^2 - d_2z. \end{aligned}$$

Now, let us consider $\sigma \in \mathbb{R}^+$. Therefore,

$$\begin{aligned} \frac{dW}{dt} + \sigma W &= \frac{r_1x}{1+fz} - mx^2 + (c_2-c_1)\frac{xz}{b+x} - d_1x + r_2y\left(1-\frac{y}{k}\right) - \mu z^2 - d_2z + \sigma x + \sigma y + \sigma z \\ &\leq x(r_1 - d_1 + \sigma - mx) - (c_1-c_2)\frac{xz}{b+x} + y\left(r_2 + \sigma - \frac{r_2}{k}y\right) + z(\sigma - d_2 - \mu z). \end{aligned}$$

If $c_1 \geq c_2$, then the above inequality reduces to the following form

$$\frac{dW}{dt} + \sigma W \leq x(r_1 - d_1 + \sigma - mx) + y\left(r_2 + \sigma - \frac{r_2}{k}y\right) + z(\sigma - d_2 - \mu z). \tag{3.2}$$

Now, the maximum value of the functions $x(r_1 - d_1 + \sigma - mx)$, $y\left(r_2 + \sigma - \frac{r_2}{k}y\right)$ and $z(\sigma - d_2 - \mu z)$ are $\frac{(r_1+\sigma-d_1)^2}{4m}$, $\frac{k(r_2+\sigma)^2}{4r_2}$ and $\frac{(\sigma-d_2)^2}{4\mu}$ respectively. So, inequality (3.2) becomes

$$\frac{dW}{dt} + \sigma W \leq G, \quad \text{where} \quad G = \frac{(r_1+\sigma-d_1)^2}{4m} + \frac{k(r_2+\sigma)^2}{4r_2} + \frac{(\sigma-d_2)^2}{4\mu}.$$

Therefore, all the solutions of the model system (2.1) lies in the interval as bellow:

$$0 \leq W \leq \frac{G}{\sigma} + G(x_0, y_0, z_0)e^{-\sigma t}.$$

Hence, each solutions of the system (2.1) have satisfied the bounded criteria. ■

3.3. *Condition of Persistence*

Before going to derive the parametric restrictions for persistence of the model system (2.1), we have considered the lemma which as bellow:

Lemma 3.1. *Let* $\zeta,\ \eta,\ X(0)$ *is strictly greater than zero. Now if* $\frac{dX}{dt} \leq X(t)\big(\zeta - \eta X(t)\big)$, *then* $\limsup_{t\to\infty} X(t) \leq \frac{\zeta}{\eta}$. *Also if* $\frac{dX}{dt} \geq X(t)\big(\zeta - \eta X(t)\big)$, *then* $\liminf_{t\to\infty} X(t) \geq \frac{\zeta}{\eta}$.

Theorem 3.3. *The model system* (2.1) *is permanent if the following properties have satisfied:*

(i) $r_1 > (1 + f\alpha_3)\left(\delta_1 + d_1 + \frac{c_1\alpha_3}{b}\right)$,

(ii) $G > \sigma(\alpha_1 + \alpha_3)$,

(iii) $c_3\alpha_1 > bd_2$,

(iv) $r_2 > \delta_2$,

(v) $c_2\alpha_1' > d_2(\alpha_1 + b)$,

where $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_1',\ \alpha_2'$ *and* α_3' *are help to the proof of this theorem.*

Proof. The 1^{st} equation of the system (2.1) can be reduces to

$$\frac{dx}{dt} \leq x(r_1 - \delta_1 - d_1 - mx).$$

Therefore, with the help of Lemma (3.1), we have $\limsup_{t\to\infty} x(t) \leq \alpha_1$ where $\alpha_1 = \frac{r_1-(\delta_1+d_1)}{m}$.
Now, from the 3^{rd} equation, we have

$$\frac{dz}{dt} \leq z\left(\frac{c_3\alpha_1}{b} - d_2 - \mu z\right).$$

Therefore, with the help of Lemma (3.1), we can write $\limsup_{t\to\infty} z(t) \leq \alpha_3$ where $\alpha_3 = \frac{c_3\alpha_1 - bd_2}{\mu b}$.
From Theorem (3.2), we get

$$\begin{aligned}
& x(t) + y(t) + z(t) \leq \frac{G}{\sigma} + G(x_0, y_0, z_0)e^{-\sigma t} \\
& \text{i.e. } \lim_{t\to\infty}[x(t) + y(t) + z(t)] \leq \frac{G}{\sigma} \\
& \text{i.e. } \limsup_{t\to\infty}\, [x(t) + y(t) + z(t)] \leq \frac{G}{\sigma} \\
& \text{i.e. } \limsup_{t\to\infty} y(t) \leq \frac{G - \sigma(\alpha_1 + \alpha_3)}{\sigma} = \alpha_2 \text{ (say)}
\end{aligned}$$

Again, the 1^{st} equation of (2.1) can be stated as

$$\frac{dx}{dt} \geq x\left(\frac{r_1}{1 + f\alpha_3} - \delta_1 - d_1 - \frac{c_1\alpha_3}{b} - mx\right).$$

Therefore, by using Lemma (3.1), we get $\liminf\limits_{t\to\infty} x(t) \geq \alpha_1'$ where $\alpha_1' = \frac{1}{m}\left\{\frac{r_1}{1+f\alpha_3} - \left(\delta_1 + d_1 + \frac{c_1\alpha_3}{b}\right)\right\}$.
Similarly,the 2^{nd} equation of the system (2.1) expressed in inequality form as bellow:

$$\frac{dy}{dt} \geq y\left(r_2 - d_2 - \frac{r_2}{k}y\right).$$

Therefore, by using Lemma (3.1), we get $\liminf\limits_{t\to\infty} y(t) \geq \alpha_2'$ where $\alpha_2' = \frac{k(r_2-d_2)}{r_2}$.
Now, the last equation of the system (2.1) reduces to in form:

$$\frac{dz}{dt} \geq z\left(\frac{c_2\alpha_1'}{b+\alpha_1} - d_2 - \mu z\right).$$

Therefore, by using Lemma (3.1), we get $\liminf\limits_{t\to\infty} z(t) \geq \alpha_3'$ where $\alpha_3' = \frac{1}{\mu}\left[\frac{c_2\alpha_1'}{b+\alpha_1} - d_2\right]$.
The system (2.1) will be persistence in a region only when all the corresponding upper and lower bound will be positive, so let us impose the following conditions:

(i) α_1 and α_1' will be positive for $r_1 > (1 + f\alpha_3)\left(\delta_1 + d_1 + \frac{c_1\alpha_3}{b}\right)$.

(ii) $\alpha_2 > 0$ if $G > \sigma(\alpha_1 + \alpha_3)$.

(iii) $\alpha_3 > 0$ if $c_3\alpha_1 > bd_2$.

(iv) $\alpha_2' > 0$ if $r_2 > \delta_2$.

(v) $\alpha_3' > 0$ if $c_2\alpha_1' > d_2(\alpha_1 + b)$.

Thus, all the conditioned are satisfied. ■

4. Stability Analysis

4.1. Equilibria and their Existence

The system has three possible positive equilibrium points and they are

(i) The trivial point of equilibria $E_0(0, 0, 0)$,

(ii) The planner equilibrium point $E_1(x_1, y_1, 0)$ where x_1 can be obtained from the following equation

$$mx^2 - (r_1 - d_1 - \delta_1)x - \delta_2 y_1 = 0, \tag{4.1}$$

and y_1 can be obtained from the following equation

$$r_2y^2 - k(r_2 - \delta_2)y - k\delta_1 x_1 = 0, \tag{4.2}$$

From the equation (4.1), we get $x_1 = \frac{(r_1-\delta_1-d_1)+\sqrt{(r_1-\delta_1-d_1)^2+4m\delta_2y_1}}{2m}$ exists if $r_1 > \delta_1 + d_1$ and from the equation (4.2), we get $y_1 = \frac{k(r_2-\delta_2)+\sqrt{k^2(r_2-\delta_2)^2+4kr_2\delta_1x_1}}{2m}$ exists if $r_2 > \delta_2$.

(iii) The interior steady state $E_*(x_*, y_*, z_*)$ where

$$z_* = \frac{c_2x_*}{\mu(b + x_*)} - \frac{d_2}{\mu} = A \text{ (say) exists if } c_2x_* > d_2(b + x_*), \tag{4.3}$$

and x_* can be obtained from the following equation

$$A_{11}x^3 + A_{12}x^2 + A_{13}x + A_{14} = 0, \tag{4.4}$$

where

$$\begin{aligned}
A_{11} &= m(1 + fA) > 0,\\
A_{12} &= mb + mbfA + \delta_1 + 2fA\delta_1 + d_1 - r_1,\\
A_{13} &= b\delta_1 + bfA\delta_1 + Ac_1 + fA^2c_1 + bd_1 + bfAd_1 - br_1 - \delta_2B - \delta_2fAB,\\
A_{14} &= -\delta_2B(b + bfA) < 0.
\end{aligned}$$

Since A_{11} is positive and A_{14} is negative quantity, so from Descarte's law of sign, equation (4.4) will have at least one real positive root, say $x^* = C$. Now, from the second equation of (2.1), we get

$$r_2y^2 - k(r_2 - \delta_2)y + k\delta_1C = 0, \tag{4.5}$$

Equation (4.5) will have a positive root for $r_2 > \delta_2$. So, the criteria of existence of the interior steady state are $c_2x^* > d_2(b + x^*)$ and $r_2 > \delta_2$.

4.2. *Local Stability Analysis*

For investigate the local stability criteria of each steady states, we have derived the Jacobian matrix as follows:

$$J(x,y,z) = \begin{pmatrix} \frac{r_1}{1+fz} - 2mx - \delta_1 - \frac{c_1bz}{(b+x)^2} - d_1 & \delta_2 & -\frac{r_1fx}{(1+fz)^2} - \frac{c_1x}{b+x} \\ \delta_1 & r_2 - \frac{2r_2}{k}y - \delta_2 & 0 \\ \frac{c_2bz}{(b+x)^2} & 0 & \frac{c_2x}{b+x} - 2\mu z - d_2 \end{pmatrix},$$

Theorem 4.1. *The trivial steady state* $E_0(0,0,0)$ *will be locally asymptotically stable if* $r_1 < \delta_1 + d_1$ *and* $r_2 < \delta_2$*, otherwise it is unstable.*

Proof. Now,

$$J(E_0) = \begin{pmatrix} r_1 - \delta_1 - d_1 & \delta_2 & 0 \\ \delta_1 & r_2 - \delta_2 & 0 \\ 0 & 0 & -d_2 \end{pmatrix},$$

It is obvious one eigen value is $-d_2$ and other two eigen values are $(r_1 - \delta_1 - d_1)$ & $(r_2 - \delta_2)$. So the model system will be asymptotically stable if $r_1 < \delta_1 + d_1$ and $r_2 < \delta_2$, otherwise it is unstable. ■

Theorem 4.2. *The planner equilibrium point* $E_1(x_1, y_1, 0)$ *must be asymptotically stable if*

(i) $r_1 < 2mx_1 + \delta_1 + d_1$,

(ii) $r_2\left(1 - \frac{2y_1}{k}\right) < \delta_2$ *and*

(iii) $\frac{c_2x_1}{b+x_1} < d_2$,

and it is unstable otherwise.

Proof. The characteristic equation of the above variational matrix at $E_1(x_1, y_1, 0)$ is

$$(\lambda - r_1 + 2mx_1 + \delta_1 + d_1)\left(\lambda - r_2 + \frac{2r_2}{k}y_1 + \delta_2\right)\left(\lambda - \frac{c_2x_1}{b+x_1} - d_2\right) = 0.$$

This implies that the planner equilibrium point will be stable if $r_1 < 2mx_1 + \delta_1 + d_1$, $r_2(1 - \frac{2y_1}{k}) < \delta_2$ and $\frac{c_2x_1}{b+x_1} < d_2$ otherwise it is unstable. ■

Theorem 4.3. *The system* (2.1) *displays asymptotically stable behaviour(LAS) near to the interior steady state* $E_*(x_*, y_*, z_*)$ *if* $M_{11} > 0$, $M_{13} > 0$ *and* $M_{11}M_{12} > M_{13}$ *hold.*

Proof. The equation of characteristic of the calculated variational matrix corresponding to the interior steady state E_* is

$$\lambda^3 + M_{11}\lambda^2 + M_{12}\lambda + M_{13} = 0, \tag{4.6}$$

where

$$\begin{aligned} M_{11} &= -\frac{r_1}{1+fz_*} + \frac{c_1bz_*}{(b+x_*)^2} + 2mx_* + \delta_1 + d_1 - r_2 + \frac{2r_2}{k}y_* + \delta_2 - \frac{c_2x_*}{b+x_*} + 2\mu z_* + d_2, \\ M_{12} &= \left(\frac{r_1}{1+fz_*} - \frac{c_1bz_*}{(b+x_*)^2} - 2mx_* - \delta_1 - d_1\right)\left(r_2 - \frac{2r_2}{k}y_* - \delta_2 + \frac{c_2x_*}{b+x_*} - 2\mu z_* - d_2\right) \\ &\quad + \left(r_2 - \frac{2r_2}{k}y_* - \delta_2\right)\left(\frac{c_2x_*}{b+x_*} - 2\mu z_* - d_2\right) + \frac{c_2bz_*}{(b+x_*)^2} - \delta_1\delta_2, \\ M_{13} &= \left(-\frac{r_1}{1+fz_*} + \frac{c_1bz_*}{(b+x_*)^2} + 2mx_* + \delta_1 + d_1\right)\left(r_2 - \frac{2r_2}{k}y_* - \delta_2\right)\left(\frac{c_2x_*}{b+x_*} - 2\mu z_* - d_2\right) \\ &\quad + \delta_1\delta_2\left(\frac{c_2x_*}{b+x_*} - 2\mu z_* - d_2\right) - \frac{c_2bz_*}{(b+x_*)^2}\left(\frac{r_1+x_*}{(1+fz_*)^2} + \frac{c_1x_*}{b+x_*}\right)\left(r_2 - \frac{2r_2}{k}y_* - \delta_2\right). \end{aligned}$$

Using Routh−Hurwitz scheme, equation (4.6) will have negative root or with have negative real parts if $M_{11} > 0$, $M_{13} > 0$ and $M_{11}M_{12} > M_{13}$ hold. Hence the theorem. ■

5. Local Bifurcation Analysis

5.1. Saddle-Node Bifurcation

Theorem 5.1. *The system* (2.1) *goes to a Saddle-Node bifurcation w. r. t. the intra-specific competition rate* μ *of predator at* $\mu = \mu_{SN} = \frac{2c_1 f d_2 + c_1 d_2}{r_1 f(b+x^*)}$ *if the following two conditions hold:*

(i) $\frac{c_2 x^*}{2(b+x^*)} - d_2 = 0,$

(ii) $\frac{r_1 f^2 x^*}{(1+fz^*)^3} - b\mu \neq 0.$

Proof. Using Sotomayor's theorem we must find Under which criterion's our proposed system (2.1) plays Saddle-Node bifurcation. The variational matrix at the interior steady state E_* will have a zero eigenvalue if

$$-\frac{r_1 f x_*}{(1+fz_*)^2} - \frac{c_1 x_*}{b+x_*} = 0, \tag{5.1}$$

$$\frac{c_2 x}{b+x} - 2\mu z - d_2 = 0. \tag{5.2}$$

From (5.1) and (5.2), we obtain the critical value (i.e. at which LP bifurcation occurs) of predator intra-specific competition rate μ as follows

$$\mu = \mu_{SN} = \frac{c_1 d_2 (2f+1)}{r_1 f(b+x_*)}.$$

Now, let $V = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $W = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ be the corresponding eigenvectors of the Jacobian matrix J and J^T respectively.

After some algebraic manipulation, we obtain $V = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $W = \begin{pmatrix} T_1 \\ \delta_2 \\ T_2 \end{pmatrix}$,

where

$$T_1 = \delta_2 - \frac{2r_2}{k} y_* - r_2,$$

$$T_2 = -\frac{[(r_1 - 2mx_* - \delta_1 - \frac{c_1 b z_*}{(b+x_*)^2} - d_1)a + \delta_1 \delta_2](b+x_*)^2}{c_2 b z_*}.$$

Now,

$$F_\mu(E_*, \mu_{SN}) = \begin{pmatrix} 0 \\ 0 \\ -(z_*)^2 \end{pmatrix}$$

So

$$W^T F_\mu(E_*, \mu_{SN}) = -T_2 (z_*)^2 \neq 0,$$

$$W^T DF_\mu(E_*, \mu_{SN}) V = \frac{6d_2}{\mu} \neq 0,$$

$$W^T D^2 F_\mu(E_*, \mu_{SN})(V\ V) = \frac{2r_1 f^2 x_*}{(1+fz_*)^3} - 2b\mu \neq 0 \quad \text{if} \frac{2r_1 f^2 x_*}{(1+fz_*)^3} - 2b\mu \neq 0$$

■

5.2. *Hopf-bifurcation analysis with its Direction*

5.2.1. *Analysis of Hopf-bifurcation*

Here, we have observed the criterion's of Hopf-bifurcation close to the interior steady state E_* of the system (2.1).

Theorem 5.2. *The system* (2.1) *exhibits hopf-bifurcation near the interior steady state* E_* *when the fear factor parameter* f *just crosses a threshold value* $f = f^*$. *The iff conditions for occurring hopf-bifurcation at* $f = f^*$ *are as follows:*

(a) $\xi_1(f^*)\xi_2(f^*) - \xi_3(f^*) = 0,$

(b) Transversality condition, $\frac{d}{df}\Big[Re(\lambda(f))\Big]_{f=f^*} \neq 0,$

where λ *is the eigenvalue of the variational matrix* J *of the model system* (2.1) *at* E_*.

Proof. The characteristic equation of (4.6) of the Jacobian matrix J of the model system (2.1) corresponding to the interior steady state E_* for the value $f = f^*$ becomes

$$(\lambda^2 + \xi_2)(\lambda + \xi_1) = 0. \tag{5.3}$$

The solutions of the above equation are $\lambda_1 = i\sqrt{\xi_2}$, $\lambda_2 = -i\sqrt{\xi_2}$ and $\lambda_3 = -\xi_1$.

For $f \in (f^* - \delta,\ f^* + \delta)$, here δ is a $+ve$ real number, the roots in general can be taken as bellow:

$$\begin{aligned}
\lambda_1(f) &= \zeta_1(f) + i\zeta_2(f),\\
\lambda_2(f) &= \zeta_1(f) - i\zeta_2(f),\\
\lambda_3(f) &= -\xi_1.
\end{aligned}$$

For investigating the transversality condition, let us substitute $\lambda_1(f) = \zeta_1(f) + i\zeta_2(f)$ in the equation (5.3) and differentiate it with respect to f, we have

$$\begin{cases}
E_1(f)\zeta_1'(f) - E_2(f)\zeta_2'(f) + G(f) = 0,\\
E_2(f)\zeta_1'(f) + E_1(f)\zeta_2'(f) + H(f) = 0,
\end{cases} \tag{5.4}$$

where

$$\begin{aligned}
E_1(f) &= 3\zeta_1^2(f) + 2\xi(f)\zeta(f) + \xi(f) - 3\zeta^2(f),\\
E_2(f) &= 6\zeta_1(f)\zeta_2(f) + 2\xi_1(\xi)\zeta_2(f),\\
G(f) &= \zeta_1^2(f)\xi_1'(f) + \xi_2'(f)\zeta_1(f) + \xi_3'(f) - \xi_1'(f)\zeta_2^2(f),\\
H(f) &= 2\zeta_1(f)\zeta_2(f)\xi_1'(f) + \xi_2'(f)\zeta_2(f).
\end{aligned}$$

Since $\zeta_1(f^*) = 0,\ \zeta_2(f^*) = \sqrt{\xi_2(f^*)}$, thus we have

$$\begin{aligned}
E_1(f^*) &= -2\xi_2(f^*),\\
E_2(f^*) &= 2\xi_1(f^*)\sqrt{\xi_2(f^*)},\\
G(f^*) &= \xi_3'(f^*) - \xi_1'(f^*)\xi_2(f^*),\\
H(f^*) &= \xi_2'(f^*)\sqrt{\xi_2(f^*)}.
\end{aligned}$$

Now, from (5.4), we get $\zeta_1(f) = -\frac{E_2(f)H(f)+E_1(f)G(f)}{E_1^2(f)+E_2^2(f)}$.
Therefore,

$$\begin{aligned}
&\frac{d}{df}\Big[Re(\lambda(f))\Big]_{f=f^*}\\
&= -\frac{E_2(f^*)H(f^*) + E_1(f^*)G(f^*)}{E_1^2(f^*) + E_2^2(f^*)}\\
&= -\frac{2\xi_1(f^*)\sqrt{\xi_2(f^*)}\xi_2'(f^*)\sqrt{\xi_2(f^*)} + \Big(-2\xi_2(f^*)\Big)\Big\{\xi_3'(f^*) - \xi_1'(f^*)\xi_2(f^*)\Big\}}{(-2\xi_2(f^*))^2 + \Big\{(2\xi_1(f^*)\sqrt{\xi_2(f^*)}\Big\}^2}\\
&= -\frac{\xi_1(f^*)\xi_2'(f^*) - \xi_3'(f^*) + \xi_1'(f^*)\xi_2(f^*)}{2\,\{\xi_2(f^*) + \xi_1(f^*)\}^2}\\
&\neq 0, \quad \text{if } \xi_3'(f^*) \neq \xi_1(f^*)\xi_2'(f^*) + \xi_1'(f^*)\xi_2(f^*) \text{ and } \lambda_3(f^*) = -\xi_1(f^*) \neq 0.
\end{aligned}$$

Thus, the condition of transversality is satisfied and occurrence at $f = f^*$ of Hopf-bifurcation whenever $\xi_3'(f^*) \neq \xi_1(f^*)\xi_2'(f^*) + \xi_1'(f^*)\xi_2(f^*)$ holds. ■

In the same way, also we investigate the conditions of Hopf-Bifurcation for the predator intra-specific competition rate μ.

5.3. Stability of Hopf-Bifurcation and its Direction

In this subsection, analytically we have examined the stability of hopf-bifurcation and its direction with respect to fear parameter f. In this regard, we have used the basic normal form of theory as indicated in [55]. At first, we find the eigenvectors v_1 and v_3 corresponding to the eigenvalues $\lambda_1 = i\omega$ and $\lambda_3 = -\xi_1$ respectively at $f = f^*$ where $\omega = \sqrt{\xi_2}$.

Let

$$v_1 = \begin{pmatrix} u_{11} - iu_{12} \\ \\ u_{21} - iu_{22} \\ \\ u_{31} - iu_{32} \end{pmatrix},$$

and

$$v_3 = \begin{pmatrix} u_{13} \\ \\ u_{23} \\ \\ u_{33} \end{pmatrix},$$

where $u_{11} = -\frac{\omega^2(P+Q)+L(SG-QP+\omega^2)+k(IQ-FG)}{N}$, $u_{12} = \frac{\omega(SG-QP+\omega^2-RP-QR+KI)}{N}$, $u_{21} = IQ - FG$, $u_{22} = -\omega I$, $u_{31} = SG - QP + \omega^2$, $u_{32} = \omega(P + Q)$, $u_{13} = \frac{(R+\xi_1)\{SG-QP-\xi_1(Q+P+\xi_1)\}-K(IQ+IG-FG)}{N}$, $u_{23} = I(Q + G) - FG$, $u_{33} = SG - GP - \xi_1(Q + P + G)$.

Now, we consider the following transformation,

$$\begin{aligned} x &= x^* + u_{11}x_1 + u_{12}y_1 + u_{13}z_1, \\ y &= y^* + u_{21}x_1 + u_{22}y_1 + u_{23}z_1, \\ z &= z^* + u_{31}x_1 + u_{32}y_1 + u_{33}z_1, \end{aligned}$$

and the above system of equation can be written as

$$\begin{cases} \frac{dx_1}{dt} = L_1, \\ \\ \frac{dy_1}{dt} = L_2, \\ \\ \frac{dz_1}{dt} = L_3, \end{cases} \tag{5.5}$$

where

$$\begin{aligned} L_1 &= \frac{A_1N_1 + (u_{11}A_2 - u_{21}A_1)(u_{13}G_1 - u_{12}G_4) + (u_{31}A_2 - u_{21}A_3)(u_{12}G_3 + u_{13}G_2)}{u_{11}\, N_1}, \\ L_2 &= \frac{G_4(u_{11}A_2 - u_{21}A_1) + G_3(u_{31}A_2 - u_{21}A_3)}{N_1}, \\ L_3 &= \frac{G_1(u_{21}A_1 - u_{11}A_2) - G_2(u_{31}A_2 - u_{21}A_3)}{N_1}, \end{aligned}$$

with $N_1 = G_1G_3 - G_2G_4,\ G_1 = u_{22}u_{31} - u_{21}u_{32},\ G_2 = u_{12}u_{21} - u_{11}u_{22},\ G_3 = u_{13}u_{21} - u_{11}u_{23},\ G_4 = u_{31}u_{23} - u_{21}u_{23},$

$$\begin{aligned}
A_1 &= \frac{r_1(x^* + u_{11}x_1 + u_{12}y_1 + u_{13}z_1)}{1 + f(z^* + u_{31}x_1 + u_{32}y_1 + u_{33}z_1)} - m(x^* + u_{11}x_1 + u_{12}y_1 + u_{13}z_1)^2 \\
&\quad - \delta_1(x^* + u_{11}x_1 + u_{12}y_1 + u_{13}z_1) - d_1(x^* + u_{11}x_1 + u_{12}y_1 + u_{13}z_1) + \delta_2(y^* + u_{21}x_1 + u_{22}y_1 + u_{23}z_1) \\
&\quad - \frac{c_1(x^* + u_{11}x_1 + u_{12}y_1 + u_{13}z_1)(z^* + u_{31}x_1 + u_{32}y_1 + u_{33}z_1)}{b + x^* + u_{11}x_1 + u_{12}y_1 + u_{13}z_1}, \\
A_2 &= r_2(y^* + u_{21}x_1 + u_{22}y_1 + u_{23}z_1) - \frac{r_2}{k}(y^* + u_{21}x_1 + u_{22}y_1 + u_{23}z_1)^2 + \delta_1(x^* + u_{11}x_1 + u_{12}y_1 + u_{13}z_1) \\
&\quad - \delta_2(y^* + u_{21}x_1 + u_{22}y_1 + u_{23}z_1), \\
A_3 &= \frac{c_2(x^* + u_{11}x_1 + u_{12}y_1 + u_{13}z_1)(z^* + u_{31}x_1 + u_{32}y_1 + u_{33}z_1)}{b + x^* + u_{11}x_1 + u_{12}y_1 + u_{13}z_1} - \mu(z^* + u_{31}x_1 + u_{32}y_1 + u_{33}z_1)^2 \\
&\quad - d_2(z^* + u_{31}x_1 + u_{32}y_1 + u_{33}z_1).
\end{aligned}$$

Obviously, $(0,0,0)$ is one steady state of the newly mention system (5.5). Also, this steady state $(0,0,0)$ of our proposed system (2.1). Therefore, the Jacobian matrix of (5.5) is as bellow:

$$J(x_1, y_1, z_1) = \begin{pmatrix} \frac{\partial L_1}{\partial x_1} & \frac{\partial L_1}{\partial y_1} & \frac{\partial L_1}{\partial z_1} \\ \frac{\partial L_2}{\partial x_1} & \frac{\partial L_2}{\partial y_1} & \frac{\partial L_2}{\partial z_1} \\ \frac{\partial L_3}{\partial x_1} & \frac{\partial L_3}{\partial y_1} & \frac{\partial L_3}{\partial z_1} \end{pmatrix},$$

where $\frac{\partial L_1}{\partial x_1} = \frac{\partial L_2}{\partial y_1} = \frac{\partial L_1}{\partial z_1} = \frac{\partial L_3}{\partial x_1} = \frac{\partial L_3}{\partial y_1} = \frac{\partial L_2}{\partial z_1} = 0, -\frac{\partial L_1}{\partial y_1} = \frac{\partial L_2}{\partial x_1} = \omega$ and $\frac{\partial L_3}{\partial z_1} = G_5$ (say).

Here, we have to calculate the terms $q_{11},\ q_{02},\ q_{20},\ M_{101},\ M_{110},\ M_{21},\ n_{11},\ n_{20},\ \omega,\ \omega_{20},\ \omega_{11}$, and q_{21} by utilizing the following relations:

$$\begin{aligned}
q_{11} &= \frac{1}{4}\left[\left(\frac{\partial^2 L_1}{\partial x_1^2} + \frac{\partial^2 L_2}{\partial y_1^2}\right) + i\left(\frac{\partial^2 L_2}{\partial x_1^2} + \frac{\partial^2 L_1}{\partial y_1^2}\right)\right], \\
q_{02} &= \frac{1}{4}\left[\left(\frac{\partial^2 L_1}{\partial x_1^2} - \frac{\partial^2 L_1}{\partial y_1^2} - 2\frac{\partial^2 L_2}{\partial x_1 \partial y_1}\right) + i\left(\frac{\partial^2 L_2}{\partial x_1^2} - \frac{\partial^2 L_2}{\partial y_1^2} + 2\frac{\partial^2 L_1}{\partial x_1 \partial y_1}\right)\right], \\
q_{20} &= \frac{1}{4}\left[\left(\frac{\partial^2 L_1}{\partial x_1^2} - \frac{\partial^2 L_1}{\partial y_1^2} + 2\frac{\partial^2 L_2}{\partial x_1 \partial y_1}\right) + i\left(\frac{\partial^2 L_2}{\partial x_1^2} - \frac{\partial^2 L_2}{\partial y_1^2} - 2\frac{\partial^2 L_1}{\partial x_1 \partial y_1}\right)\right], \\
M_{21} &= \frac{1}{8}\left[\left(\frac{\partial^3 L_1}{\partial x_1^3} + \frac{\partial^3 L_1}{\partial x_1 \partial y_1^2} + \frac{\partial^3 L_2}{\partial x_1^2 \partial y_1} + \frac{\partial^3 L_2}{\partial y_1^3}\right) + i\left(\frac{\partial^3 L_2}{\partial x_1^3} + \frac{\partial^3 L_2}{\partial x_1 \partial y_1^2} - \frac{\partial^3 L_1}{\partial x_1^2 \partial y_1} - \frac{\partial^3 L_1}{\partial y_1^3}\right)\right], \\
\omega &= -\frac{\partial L_1}{\partial y_1}, \\
n_{11} &= \frac{1}{4}\left[\left(\frac{\partial^2 L_3}{\partial x_1^2} + \frac{\partial^2 L_3}{\partial y_1^2}\right)\right], \\
n_{20} &= \frac{1}{4}\left[\left(\frac{\partial^2 L_3}{\partial x_1^2} - \frac{\partial^2 L_3}{\partial y_1^2} - 2i\frac{\partial^2 L_3}{\partial x_1 \partial y_1}\right)\right].
\end{aligned}$$

To get the value of ω_{11} and ω_{20}, we must solve,

$$G_5\omega_{11} = -n_{11},\ (N_1 - 2i\omega)\omega_{20} = -n_{20},$$

where

$$\begin{aligned}
M_{110} &= \frac{1}{2}\left[\left(\frac{\partial^2 L_1}{\partial x_1 \partial z_1} + \frac{\partial^2 L_2}{\partial y_1 \partial z_1}\right) + i\left(\frac{\partial^2 L_2}{\partial x_1 \partial z_1} - \frac{\partial^2 L_1}{\partial y_1 \partial z_1}\right)\right], \\
M_{101} &= \frac{1}{2}\left[\left(\frac{\partial^2 L_1}{\partial x_1 \partial z_1} - \frac{\partial^2 L_2}{\partial y_1 \partial z_1}\right) + i\left(\frac{\partial^2 L_2}{\partial x_1 \partial z_1} + \frac{\partial^2 L_1}{\partial y_1 \partial z_1}\right)\right], \\
q_{21} &= M_{21} + 2M_{110}\omega_{11} + M_{101}\omega_{20}.
\end{aligned}$$

Using the above results, we examine the value of the following expressions,

$$
\begin{aligned}
C_1(0) &= \frac{i}{2\omega}\left(q_{20}q_{11} - 2|q_{11}|^2 - \frac{|q_{02}|^2}{3}\right) + \frac{q_{21}}{2},\\
\mu_2 &= -\frac{Re\{C_1(0)\}}{\alpha'(0)},\\
\beta_2 &= 2Re\{C_1(0)\},\\
T_2 &= -\frac{Im\{C_1(0)\} + \mu_2\omega'(0)}{\omega},
\end{aligned}
$$

where $\alpha'(0) = \frac{d}{df}[Re\{\lambda_1(f)\}]_{f=f^*}$ and $\omega'(0) = \frac{d}{df}[Im\{\lambda_1(f)\}]_{f=f^*}$. Thus, we have to reached the theorem as bellow:

Theorem 5.3. The proposed system (2.1) exhibits sub-critical Hopf-Bifurcation if $\mu_2 < 0$ and super-critical if $\mu_2 > 0$. The periodic solutions are stable if $\beta_2 < 0$ and unstable if $\beta_2 > 0$. Furthermore, bifurcating periodic solutions increase when $T_2 > 0$ and decrease when $T_2 < 0$.

In the same way we can write down the stability of Hopf-Bifurcations and its direction corresponding to predator intra specific competition rate μ.

Numerical Simulations and Discussions

In this portion, we perform meticulous numerical simulations using mathematical software Matlab (2018a) and Matcont [56] to examine, validate and verify the analytical findings of our proposed system. We have seen that the numerical simulations mainly varying the stage of predator induced fear (f) and predator intra-specific competition rate (μ). Based on real-life observations, it is authoritative to mention that it is dificult enough to choose numerical values of the set of parameters of the system. However, for the simulation purpose, we fix value of different model parameters as $\delta_1 = 0:1$; $\delta_2 = 0:15$; $d_1 = 0.01$; $d_1 = 0.03$; $c_1 = 0.2$; $c_2 = 0.1$; $b = 0.5$; $k = 10$, $\mu = 0.01$ keeping a realistic balance with the other articles [26, 47, 57–59]. Furthermore, there are enormous number of prey species in the ecological world and their rate of growth are not same. Few species grow very fast, some species accustomed with medium level of growth and few others grow slowly. Thus we simulate the model system keeping growth rate of prey species into 3 different levels namely low level, medium level and high level. The outcomes and observations of numerical simulation have been categorized and rectified into the cases which as bellow:

Case-1: The role of predator induced fear and predator intra-specific competition when growth rate of prey species is in low level

In this case we assume the rate of growth of prey spices of both territory are in low level. Here, we consider the logistic growth rate of prey population of unreserved zone as $r_1 = 0.08$ and intrinsic growth rate of prey population of reserved zone as $r_2 = 0.05$ and simulate the model system varying the rate of intra-specific competition among the predators itself and amount of fear level induced by the predator. Our observations are listed in the following two sub-cases:

Sub case-1A: Role of predator intra-specific competition rate (μ) when the predator incited fear is constant

To know the effect of predator intra specific competition rate (μ) in the dynamics of the model system, we first consider μ = 0 and investigate that the system displays oscillatory behaviour by generating limit cycle oscillations (see Fig.1A). But when the value of is increased to μ = 0.01 then the limit cycle are not appears and the model system exhibits stable behaviour as shown in Fig.1B. Thus, the system goes to a Hopf-Bifurcation w. r. t. the parameter (see Fig. 2). Clearly, from Fig. 2 it have been seen that the sys-

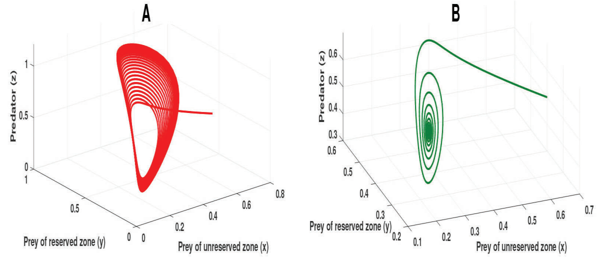


Fig. 1. Depicts the role of predator intra-specific competition rate in the system dynamics when predator induced fear is present in the system (as f = 1). Fig.A shows limit cycle oscillations for μ= 0 while Fig.B exhibits stable focus for μ = 0.01.

tem enters into a local asymptotically stable area by washing out the periodic solutions when crosses the threshold value μ = μ* = 0.001408. We have derived 1st coefficient of lyapunov (l1), which is found as l_1 = –0.004994457. This indicates that super-critical hopf-bifurcation occurs for μ = μ* = 0.001408 with orbitally stable bifurcating periodic solutions. We have also plotted Floquet Multipliers versus predator intra-specific competition rate (μ) to confirm the stability of limit cycle oscillations (see Fig.3A & Fig.3B). Here it is observed that modulus of all the multipliers are less than 1, which exhibits the occurrence of a orbitally stable limit cycle. Furthermore, with the variation of predator intra-specific competition rate, we have plotted a diagram of real parts of each eigenvalues of the corresponding Variational matrix at the hopf-point and observed that real parts of the conjugate complex eigenvalues become zero at the hopf-point and the other one remains negative (see Fig. 4A). It is also observed that the bifurcating periodic solutions are decreases with the increase of predator intra-specific competition rate (μ), which has been shown in Fig. 4B. Finally, along the critical line of Hopf-Bifurcation curve near the hopf-point, we have plotted two dimensional projection of Hopf-Bifurcation curves in distinct parametric planes to observe the inter-linkage of distinct system parameters with predator intra-specific competition rate (μ). The possible biological explanations of different diagrams of Fig.5 have been discussed below:

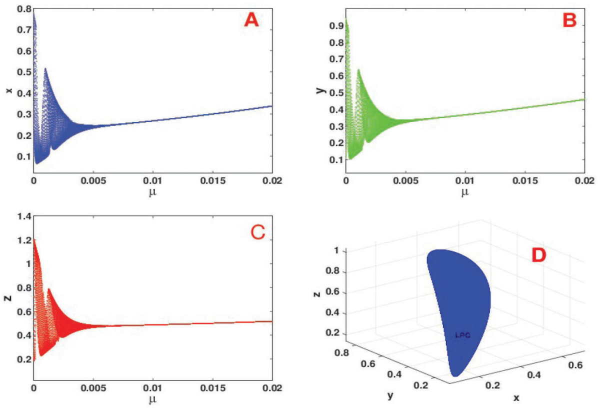


Fig. 2. Depicts the bifurcation diagram w.r.t. predator intra-specific competition rate μ. Fig. A, Fig. B and Fig. C collectively reflect that in the presence of predator induced fear (f = 1) the system exhibits limit cycle oscillations and remains unstable when μ ≤ 0.001408; otherwise it shows stable behaviour. At the bifurcation point μ = 0.001408, Fig. D represents the occurrence of a stable LPC (Limit Point Cycle).

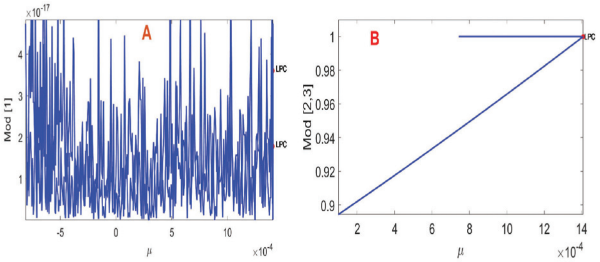


Fig. 3. Fig. A and Fig. B cooperatively show the stability of limit cycle oscillations w.r.t. predator intra-specific competition rate μ. At μ = 0.001408 as the modulus of all the Floquet multipliers remain below the level 1.

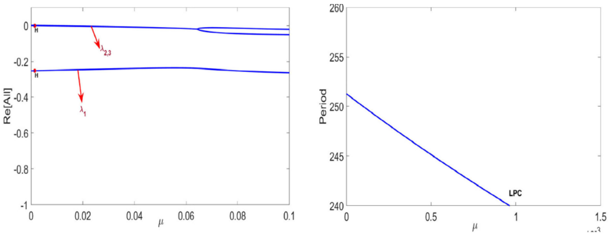


Fig. 4. Fig.A depicts the change of real part of each three eigenvalues of the corresponding Jacobi matrix near hopf point with the variation of predator intra-specific competition rate μ. Fig.B exhibits periodic status of limit cycle; in presence of predator induced fear, the period decreases with the increase of predator intra-specific competition rate μ.

Fig.5A indicates that in case of lower propagation of prey species in unreserved zone, predator's engagement in conspecific competition () must be higher to alter the state of stability of the system. Interestingly, continuous increase in the rate of intra-specific competition among the predators endures the prey species of unreserved territory to be less afraid from predation and prey species has the tendency to stay longer time in the unreserved zone; even more number of prey move from reserved to unreserved zone. As a result more number of prey species found in the unreserved territory, which indirectly helps the predators as they can consume prey species more easily. These phenomenon have been reflected in the figures Fig.5B, Fig.5E, Fig.5F & Fig.5G.With the increase of predator intra-specific competition rate , the growth factor of predator population decreases continuously because of a fraction of energy of predator species has been wasted during the clash among themselves (see Fig.5K). Moreover, from Fig.5C it is clear that increase in the predator intra-specific competition rate reduces the tendency of prey species to engage into a clash for living resources among themselves due to the accessibility of more grazing field and as a consequence the half saturation constant of prey species reduced, which is biologically meaningful. Also, it is notified that the carrying capacity for prey population in reserved zone must be within a suitable interval (here k_2 belongs to [10,18] to support the hopf-bifurcation curve.

Sub case-1B: Role of predator incited fear (f) when intra-specific competition rate is constant

In the proposed predator-prey system fear induced by the predator generally changes the dynamics of the system from its unstable state to the stable state,

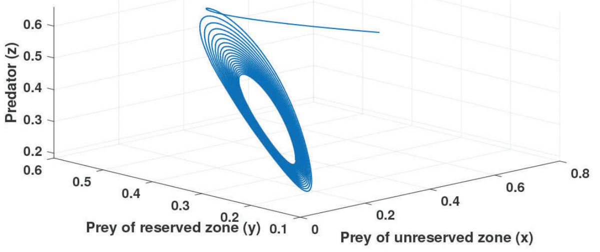


Fig. 6. Phase diagram of the model system where predator intra-specific competition rate was taken as μ = 0.001 with a very high level of fear f = 50. Phase diagram indicates that model system follows limit cycle oscillation even in presence of high level of fear.

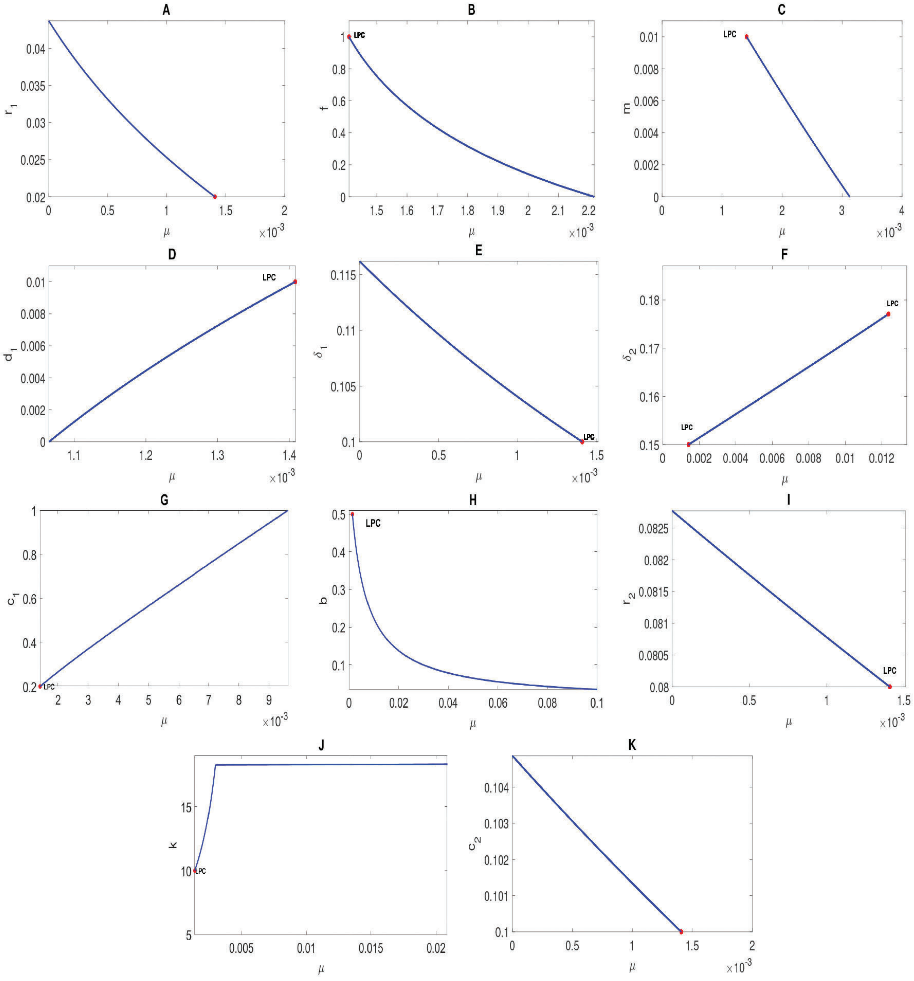
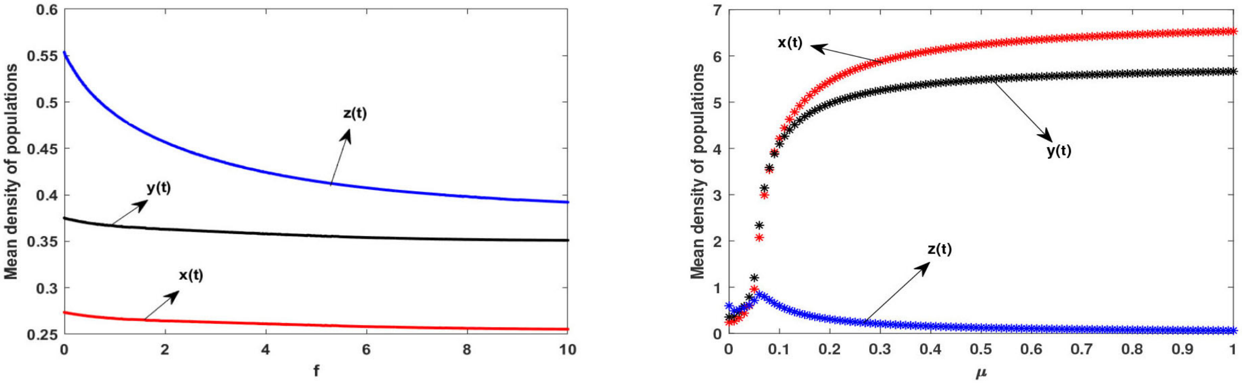


Fig. 5. 2D projection of hopf-bifurcation graphs in distinct parametric planes which reflect the inter-linkage of different model parameters with predator intra-specific competition rate (μ) along the critical line of hopf-bifurcation curve near the hopf-point.

i.e. the level of fear can behaves as a crucial stabilizing factor in the dynamics of a eco-model system although few exceptions are already cited in this regard (See [46, 60]). Here we simulate the model system from its unstable state fixing the intra-specific competition rate $\mu = 0:001$ and increase the value of the predator incited fear parameter f from 0 to 50. Here, we notice that level of fear has no effect in changing the unstable behaviour of system dynamics (See Fig.6). In Fig.7, we have drawn the graph of the mean density of each species where fear parameter f and predator intra-specific competition rate are varying separately. Fig.7(a) shows that the density of predator and density of prey in the unreserved zone decrease with the increase of level of fear whereas density of prey species in reserved zone remains at a fixed level after a slight depletion. On the other hand, from Fig. 7(b), it is observed that under continuous incremental pressure of intra-specific competition among the predators, mean density of predator population reduces while mean density of prey species in both reserved and unreserved zone increase, which is a natural phenomenon in ecology.



(a) (b)

Fig. 7. Fig.(a) and Fig.(b) exhibit the change of mean biomass of all populations w.r.t. the change of fear factor f and the rate of predator intra-specific competition respectively when the growth rate of prey population is in low level (Case I). Form Fig. (a), it have been checked that the mean density of predator species (z(t)) as well as prey species at unreserved zone(x(t)) decrease sharply whereas mean biomass of prey individuals at reserved zone (y(t)) remains at a fixed level after a slight depletion in density when fear level (f) increases. However, from Fig.(b), it is observed that under continuous increment in predator intra-specific competition rate , mean density of predator population reduces while mean density of both zone's prey species increase.

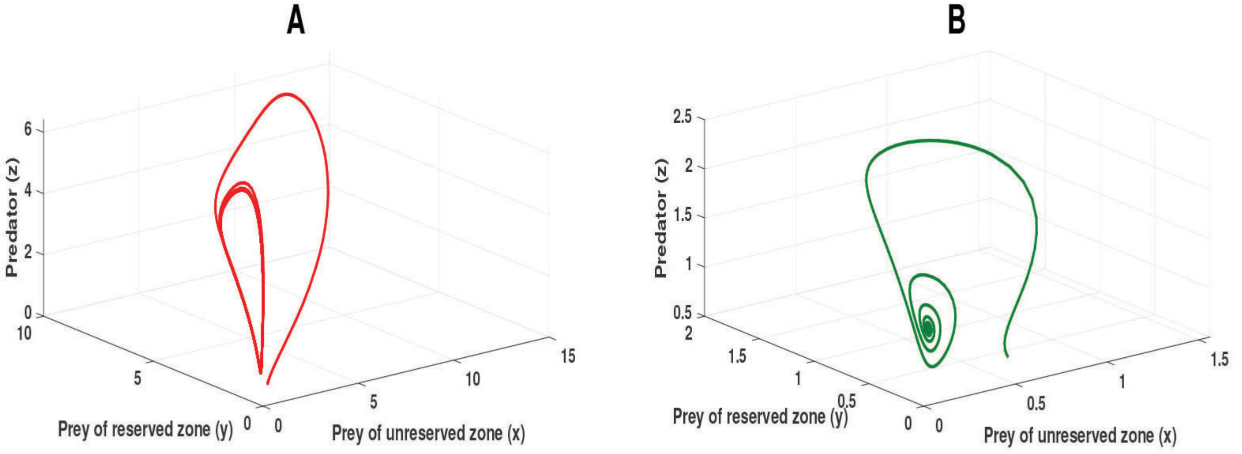


Fig. 8. Depicts the role of predator incited fear (f) when intra-specific competition rate is constant (= 0:01) and growth rate of prey species is in medium level (Sub case-2A). Fig.A shows limit cycle oscillations in the absence of fear (i.e. f = 0) while Fig.B exhibits stable focus in presence of fear, i.e. for f = 1.

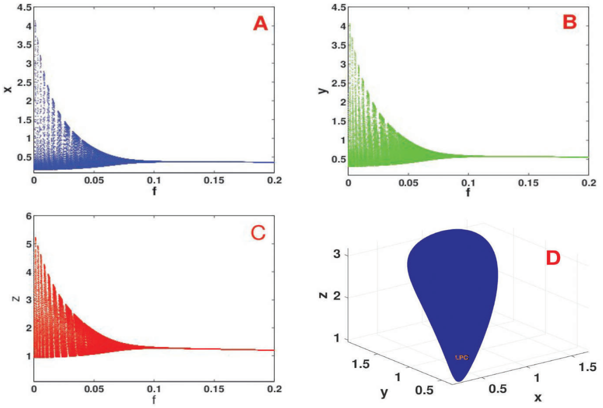


Fig. 9. Bifurcation diagram w.r.t. fear parameter f when growth of prey species is in medium level (Case-2). Fig.A, Fig.B, and Fig.C jointly exhibit that the system dynamics of the model system (2.1) unstable behaviours until f crosses the threshold parametric value f = f* = 0.065478 and further increment in f makes the system dynamics stable; while Fig.D shows the occurrence of the stable LPC (Limit Point Cycle) at the bifurcation point f = 0.065478.

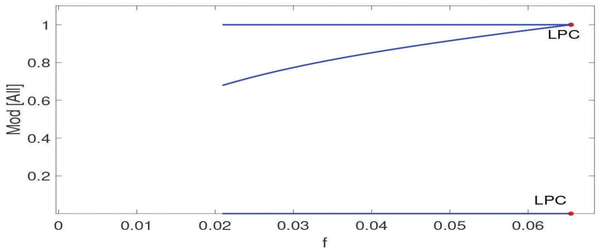


Fig. 10. Depicts the stability of limit cycle oscillations with the variation of fear parameter f. This figure exhibits that all the Floquet multipliers of the system dynamics w.r.t. f lie below 1 which shows the occurrence of a stable limit cycle oscillations.

Case-2: The role of predator induced fear and predator intra-specific competition when growth rate of prey population is in medium level

In this case we assume that the prey spices grow in moderate level. Here we consider the logistic growth rate at unreserved zone as $r_1 = 0.2$ and intrinsic growth rate at reserved zone as $r_2 = 0.09$ and simulate the model system varying the rate of intra-specific competition among the predators itself and the amount of fear induced by the predator. Our observations are listed in the following sub-cases:

Also, it is observed that with the increase of fear level induced by the predators, the period of the bifurcating limit cycle decreases from 189 to 129, which has been shown in Fig. 12. Finally, we have plotted 2D projection of hopfbifurcation curve in distinct parametric planes to observe the inter-linkage of distinct model parameters with the level of fear (f) along the critical line of hopf-bifurcation curve around the hopf-point (see Fig.13). Fig.13A reveals that for a developed posterity level of prey species of unreserved zone, predators need to prompt increased amount of fear for periodic coexistence of the populations. Fig. 13B exhibits that for an incremental level of fear (f), natural mortality of prey species of unreserved zone should be lower to support the critical line of Hopf-bifurcation curve. Fig. 13C and Fig. 13D indicate that a high level of fear leads the system to produce oscillatory dynamics if prey migration rate from unreserved to reserved zone decreases and conversion rate of prey to predator declines.

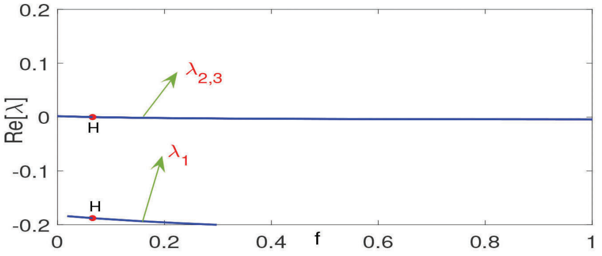


Fig. 11. Graph of real part of eigenvalues of the Jocibian matrix when the fear factor f is varying in the range [0, 1]. From the graph its clear that at the Hopf-point real part of one eigenvalue is negative and other two eigenvalues are zero

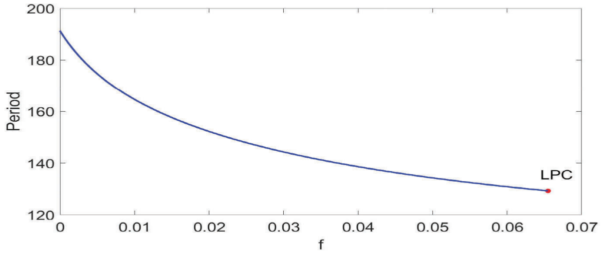


Fig. 12. Shows the relation between period of limit cycle oscillations and fear factor (f). Period of limit cycle oscillations decreases with the increase of fear factor (f).

In the case of medium level of growth of prey population, the role of intra-specific competition among the predators have been studied under two scenarios namely in presence of fear factor and in absence of fear factor, which have been reported into the following sub-cases:

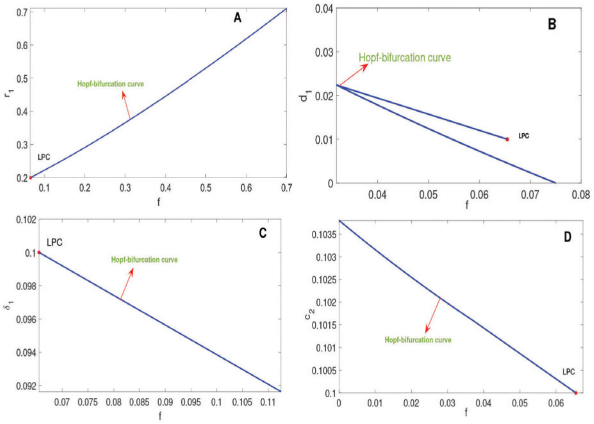


Fig. 13. 2D projection of Hopf-bifurcation curves in different parametric planes, which reflects the inter linkage between fear factor and other model parameters near the Hopf point.

Sub case-2B: Role of predator intra-specific competition rate in absence of fear factor in the system By the various aspects of numerical simulation, it has been observed that in absence of fear (or when the fear is in low level) the system undergoes two consecutive saddle-node (LP) bifurcations (see Fig. 14). The first one occurs at $\mu = 0.017111$ with normal form co-efficient $\alpha = -0.006748236$ near the interior equilibrium point E*(1.13198; 1.533767; 2.300451) and a neutral saddle point occurs at = 0:012748 around the interior equilibrium point E* (4.094831; 4.157277; 4.629028). The second LP point occurs for $\mu = 0.011165$ with normal form co-efficient $\alpha = -0.002748236$ near the interior equilibrium point E*(7.686859; 6.491159; 5.722563).

Sub case-2C: Role of predator intra-specific competition rate in presence of fear factor in the system

In presence of fear factor in the system, through extensive numerical simulation, we could not find any structural change in dynamics of the system through the variation of rate of intra-specific competition among the predators, i.e. in presence of predator incited fear the role of intra-specific competition among the predators is not significant enough to change the structure of the system dynamics. However, to explore the change of mean density of each populations, we have plotted the mean biomass of all the species where fear parameter f and predator intra-specific competition rate μ are varying separately (see Fig.15). Fig.15(a) demonstrates the fact that under continuous increment in fear level ($f \in [0; 10]$, the predator population mean biomass decreases while the mean density of the prey species of both reserved and unreserved areas remain at a constant level after a slight initial decline. On the other hand, under continuous increment of predator intraÛspecific competition rate ($\mu \in [0; 1]$), the mean density of predator species reduces and approaches to zero while the mean density of prey species of both zones increase as displayed in Fig.15(b).

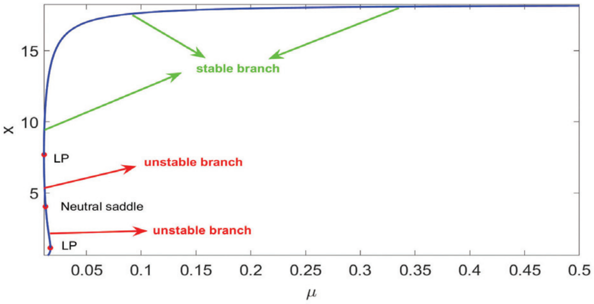
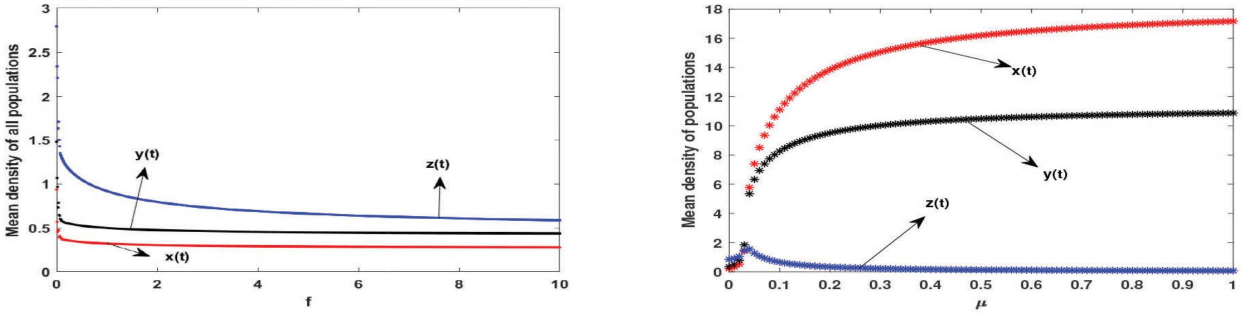


Fig. 14. Depicts the saddle-node bifurcation with respect to predator intra-specific competition rate μ. The first LP point occurs for μ= 0.017111 and the second one occurs for $\mu = 0.011165$.

Case-3: The role of predator induced fear and predator intra-specific competition when growth rate of prey species is in high level. In this case, we assume that the prey spices grows very fast. Here, we consider the logistic growth rate at unreserved zone as $r_1 = 0{:}6$ and intrinsic growth rate at reserved zone as $r_2 = 0{:}5$ and simulate the model system varying the rate of intra-specific competition among the predators and amount of fear induced by the predator. Here, through extensive numerical simulation, we could not find any structural change in dynamics of the system. Thus we can say that the fear factor and intra-specific competition among the predators are not significant enough to change the structure of the system dynamics when the growth rate of prey population is very high. However, to explore the change of mean density of all the populations, we have plotted the mean density of all the species where fear parameter f and predator intra-specific competition rate μ are varying separately (see Fig. 16). Here, we have noticed that mean density of all the populations initially reduce to a specific level and then remain constant when the level of fear is gradually increased (see Fig. 16(a)). On the other hand, Fig. 16(b) reveals that mean density of the prey species of unreserved territory increases continuously whereas the mean density of the prey species of reserved territory remains in a specific level after an initial rise and mean density of predators reduces with the increase of predator intraÛspecific competition rate μ.

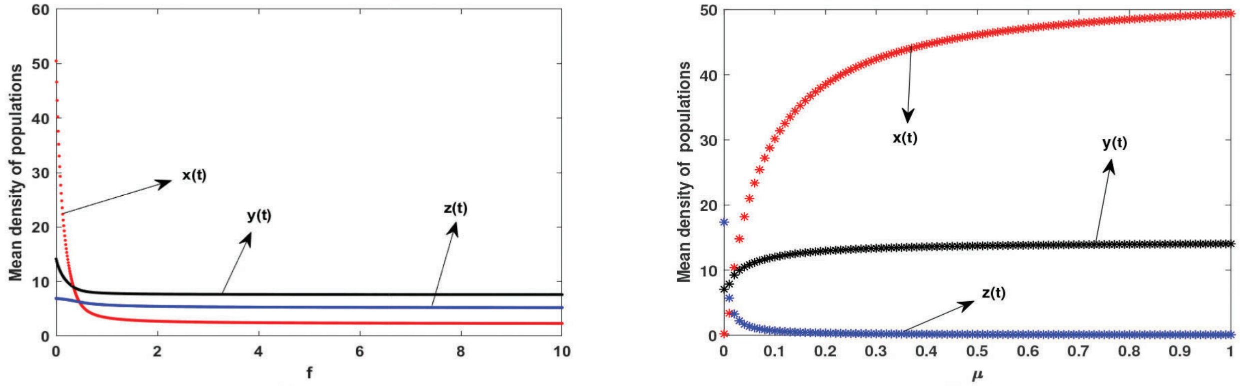
Conclusion

In this article, a prey-predator model has been considered where the habitat of prey species is partitioned into two zones, namely reserved zone and unreserved zone. Both the prey and predator species are allowed in unreserved zone to access their living resources at any time; whereas reserved zone is protected only for prey species and predators are not allowed by any means in this zone. At the unreserved zone, predation is allowed and prey species are in fear of predation by the predators. As a consequence, it is assumed that prey species in unreserved zone shows many antipredator behaviour and defense mechanism (due to fear of predation), which create the shortage of food and compels the predators to engage into a clash among themselves. We built a mathematical model and tested the positivity, boundedness, and permanence of the model system's solutions. We discovered all of the system's equilibrium points as well as its existence requirements. Stability analysis is conducted near all of the model system's equilibrium points, and the system's dynamics are thoroughly described. We used care-



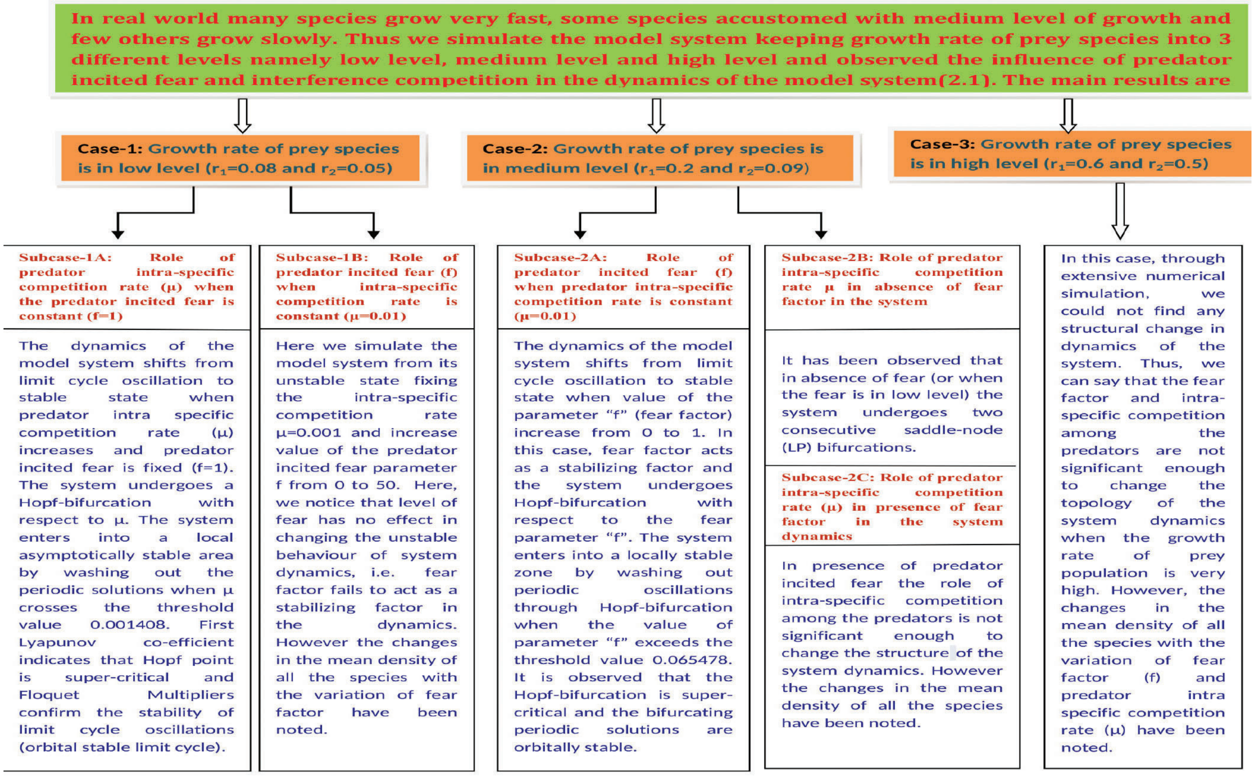
(a) (b)

Fig. 15. Exhibit the change of mean density of all the populations with the change of fear level (f) and predator intra-specific competition rate (μ) when the growth rate of prey species is in medium level. Fig. (a) demonstrates the fact that under continuous increment in fear level, the predator population mean biomass decreases while the mean density of the prey species of both reserved and unreserved areas remain at a constant level after a slight initial decline. Fig.(b) shows that mean density of predator species reduces and approaches to zero while the mean density of prey species of both zones increase when predator intra-specific competition rate gradually increase in the interval [0,1].



(a) (b)

Fig. 16. Reflects change of the mean density of all the species w.r.t. fear factor (f) and predator intra-specific competition rate () when the growth rate of prey is high. Fig.(a) reveals that mean density of all the populations initially reduce to a specific level and then remain constant when fear level is increased gradually. Fig.(b) demonstrates that mean density of the prey species of unreserved territory increases continuously whereas the mean density of the prey species of reserved territory remains in a specific level after an initial rise and mean density of predators reduces with the increase of predator intra-specific competition rate .



In real world many species grow very fast, some species accustomed with medium level of growth and few others grow slowly. Thus we simulate the model system keeping growth rate of prey species into 3 different levels namely low level, medium level and high level and observed the influence of predator incited fear and interference competition in the dynamics of the model system(2.1). The main results are

Case-1: Growth rate of prey species is in low level (r_1=0.08 and r_2=0.05)

Subcase-1A: Role of predator intra-specific competition rate (μ) when the predator incited fear is constant (f=1)

The dynamics of the model system shifts from limit cycle oscillation to stable state when predator intra specific competition rate (μ) increases and predator incited fear is fixed (f=1). The system undergoes a Hopf-bifurcation with respect to μ. The system enters into a local asymptotically stable area by washing out the periodic solutions when μ crosses the threshold value 0.001408. First Lyapunov co-efficient indicates that Hopf point is super-critical and Floquet Multipliers confirm the stability of limit cycle oscillations (orbital stable limit cycle).

Subcase-1B: Role of predator incited fear (f) when intra-specific competition rate is constant (μ=0.01)

Here we simulate the model system from its unstable state fixing the intra-specific competition rate μ=0.001 and increase value of the predator incited fear parameter f from 0 to 50. Here, we notice that level of fear has no effect in changing the unstable behaviour of system dynamics, i.e. fear factor fails to act as a stabilizing factor in the dynamics. However the changes in the mean density of all the species with the variation of fear factor have been noted.

Case-2: Growth rate of prey species is in medium level (r_1=0.2 and r_2=0.09)

Subcase-2A: Role of predator incited fear (f) when predator intra-specific competition rate is constant (μ=0.01)

The dynamics of the model system shifts from limit cycle oscillation to stable state when value of the parameter "f" (fear factor) increase from 0 to 1. In this case, fear factor acts as a stabilizing factor and the system undergoes Hopf-bifurcation with respect to the fear parameter "f". The system enters into a locally stable zone by washing out periodic oscillations through Hopf-bifurcation when the value of parameter "f" exceeds the threshold value 0.065478. It is observed that the Hopf-bifurcation is super-critical and the bifurcating periodic solutions are orbitally stable.

Subcase-2B: Role of predator intra-specific competition rate μ in absence of fear factor in the system

It has been observed that in absence of fear (or when the fear is in low level) the system undergoes two consecutive saddle-node (LP) bifurcations.

Subcase-2C: Role of predator intra-specific competition rate (μ) in presence of fear factor in the system dynamics

In presence of predator incited fear the role of intra-specific competition among the predators is not significant enough to change the structure of the system dynamics. However the changes in the mean density of all the species have been noted.

Case-3: Growth rate of prey species is in high level (r_1=0.6 and r_2=0.5)

In this case, through extensive numerical simulation, we could not find any structural change in dynamics of the system. Thus, we can say that the fear factor and intra-specific competition among the predators are not significant enough to change the topology of the system dynamics when the growth rate of prey population is very high. However, the changes in the mean density of all the species with the variation of fear factor (f) and predator intra specific competition rate (μ) have been noted.

Fig. 17. Demonstrates the whole findings of the proposed model system (2.1) in a summarised form.

ful numerical simulations to test and confirm our model system's analytical results. We have run numerical simulations mainly varying the level of predator induced fear (*f*) and predator intra-specific competition rate (μ) keeping growth rate of prey species into 3 different levels namely low level, medium level and high level. At first, we consider the scenario where the prey species grows slowly in the eco-system. In this case, the outcomes of model simulation suggest that the cost of fear does not have any contributory role to change inter-relationship of the individuals. As long as there is a fear component present, predator intra-specific competition (μ) plays a key role in regulating the system dynamics. It is noted that when fear factor is in moderate level (*f* = 1) the system undergoes a Hopf-bifurcation when the value of the parameter increases from 0 to 0:01. In fact, in this situation, the system enters into a local asymptotically stable area through super-critical Hopfbifurcation by excluding periodic solution when the predator intra-specific competition rate exceeds the threshold value 0.001408. It is also observed that the period of the bifurcating periodic solution decreases with the increase of predator intra-specific competition rate (μ). We have plotted two dimensional projection of hopf-bifurcation curve in different parametric planes to observe the inter-linkage of different model parameters with predator intra-specific competition rate (μ) along the critical line of Hopf-bifurcation curve around the Hopf-point and its possible ecological explanations are discussed.

In the second case, we have dealt with the scenario where the prey species grows in moderate rate. In this case, it is observed that the dynamics of model system shifts from limit cycle oscillation to stable state when the value of the fear factor parameter f increases from 0 to 1. Here, fear factor acts as a stabilizing factor and the system undergoes a Hopf-bifurcation with respect to the fear parameter f. The system enters into a locally stable zone by washing out periodic oscillations through Hopf-bifurcation when the value of the fear parameter f exceeds the threshold value f^* = 0.065478. Here, the Hopf-bifurcation is super-critical and the bifurcating periodic solutions are orbitally stable.

It is observed that in absence of fear (or when the amount of fear is in very low level), intra-specific competition among predators is capable to reduce fluctuation in the system and stimulates stable coexistence around a fixed equilibrium point through two consecutive LP bifurcations. But, in the presence of predator induced fear, the role of intra-specific competition among the predators is not significant enough to change the structure of the system dynamics; although the change of mean density of all the species have been noted.

In the third case, we have dealt with the scenario where the growth rate of prey species is in high level. In this case, it is observed that fear factor and intra-specific competition rate among predators have no significant role in the dynamics of the model system . However, we have noticed that mean density of all the populations initially reduces to a specific level and then remains constant when the level of fear gradually increases (see Fig.16(a)). On the other hand, the mean density of the prey species of unreserved territory increases continuously whereas the mean density of the prey species of reserved territory remains in a specific level after an initial rise and mean density of predators reduces with the increase of predator intra-specific competition rate (see Fig.16(b)). Finally, it is authoritative to mention that in this article both the migration rates from reserved zone to unreserved zone and vice versa are considered as constants. It would be more interesting to study the system dynamics of the model system treating both the migration rates as predator abundance function, which has been left for future work.

Conflict of Interest

The authors declare that they have no conflict of interest regarding the publication of this article.

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